

Solving the Ku-Wales conjecture on the eigenvalues of the derangement graph

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Abstract

We give a new recurrence formula for the eigenvalues of the derangement graph. Consequently, we provide a simpler proof of the Alternating Sign Property of the derangement graph. Moreover, we prove that the absolute value of the eigenvalue decreases whenever the corresponding partition decreases in the dominance order. In particular, this settles affirmatively a conjecture of Ku and Wales (J. of Combin. Theory, Series A 117 (2010) 289–312) regarding the lower and upper bound for the absolute values of these eigenvalues.

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1 Introduction

Let G be a finite group and S be a subset of G . The Cayley graph $\Gamma(G, S)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $uv^{-1} \in S$. We require that S is a nonempty subset of G satisfying the condition that $s \in S \implies s^{-1} \in S$ and $1 \notin S$.

The *derangement graph* Γ_n is the Cayley graph $\Gamma(\mathcal{S}_n, \mathcal{D}_n)$ where \mathcal{S}_n is the symmetric group on $[n] = \{1, \dots, n\}$, and \mathcal{D}_n is the set of derangements in \mathcal{S}_n . That is, two vertices g, h of Γ_n are joined if and only if $g(i) \neq h(i)$ for all $i \in [n]$, or equivalently gh^{-1} fixes no point.

Clearly, Γ_n is vertex-transitive, so it is D_n -regular where $D_n = |\mathcal{D}_n|$. It is well known that the largest eigenvalue of a regular graph is its degree. However, it is generally difficult to determine the smallest eigenvalue of a regular graph. Recently, after having derived a recurrence formula (see Theorem 1.2 below) for the eigenvalues of Γ_n , Renteln [8] showed that the smallest eigenvalue μ of Γ_n is $-\frac{D_n}{n-1}$. The value of μ was also determined independently by Ellis et al. [3] in their seminal work on intersecting families of permutations. The recurrence obtained by Renteln was later used by Ku and Wales [4] to prove the Alternating Sign Property (ASP) of the derangement graph (Theorem 1.3). The

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purpose of this paper is to give a new recurrence formula for these eigenvalues. This new recurrence, which follows from the property of shifted schur functions, provides a simpler proof of the ASP and settles affirmatively a conjecture of Ku and Wales regarding the lower bound and upper bounds for the absolute values of these eigenvalues.

Recall that a Cayley graph $\Gamma(G, S)$ is *normal* if S is closed under conjugation. It is well known that the eigenvalues of a normal Cayley graph $\Gamma(G, S)$ can be expressed in terms of the irreducible characters of G .

Theorem 1.1 ([1, 2, 5, 6]). *The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are integers given by*

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s), \quad (1)$$

where χ ranges over all the irreducible characters of G . Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Recall that a partition λ of n , denoted by $\lambda \vdash n$, is a weakly decreasing sequence $\lambda_1 \geq \dots \geq \lambda_r$ with $\lambda_r \geq 1$ such that $\lambda_1 + \dots + \lambda_r = n$. We write $\lambda = (\lambda_1, \dots, \lambda_r)$. The *size* of λ , denoted by $|\lambda|$, is n and each λ_i is called the *i-th part* of the partition. We also use the notation $(\mu_1^{a_1}, \dots, \mu_s^{a_s}) \vdash n$ to denote the partition where μ_i are the distinct nonzero parts that occur with multiplicity a_i . For example,

$$(5, 5, 4, 4, 2, 2, 2, 1) \longleftrightarrow (5^2, 4^2, 2^3, 1).$$

Clearly, the derangement graph Γ_n is normal since the set \mathcal{D}_n is closed under conjugation. On the other hand, it is well known that both the conjugacy classes of \mathcal{S}_n and the irreducible characters of \mathcal{S}_n are indexed by partitions λ of $[n]$. Therefore, the eigenvalue η_{χ_λ} of the derangement graph can be denoted by η_λ . Throughout, we shall use this notation.

To describe the recurrence formula of Renteln, we require some terminology. To the Young diagram of a partition λ , we assign *xy*-coordinates to each of its boxes by defining the upper-left-most box to be $(1, 1)$, with the x axis increasing to the right and the y axis increasing downwards. Then the *hook* of λ is the union of the boxes $(x', 1)$ and $(1, y')$ of the Ferrers diagram of λ , where $x' \geq 1$, $y' \geq 1$. Let \hat{h}_λ denote the hook of λ and let h_λ denote the size of \hat{h}_λ . Similarly, let \hat{c}_λ and c_λ denote the first column of λ and the size of \hat{c}_λ respectively. Note that c_λ is equal to the number of rows of λ . When λ is clear from the context, we replace \hat{h}_λ , h_λ , \hat{c}_λ and c_λ by \hat{h} , h , \hat{c} and c respectively. Let $\lambda - \hat{h} \vdash n - h$ denote the partition obtained from λ by removing its hook. Also, let $\lambda - \hat{c}$ denote the partition obtained from λ by removing the first column of its Ferrers diagram, i.e. $(\lambda_1, \dots, \lambda_r) - \hat{c} = (\lambda_1 - 1, \dots, \lambda_r - 1) \vdash n - r$.

Theorem 1.2 ([8] Renteln's Formula). *For any partition λ , the eigenvalues of the derangement graph Γ_n satisfy the following recurrence:*

$$\eta_\lambda = (-1)^h \eta_{\lambda - \hat{h}} + (-1)^{h + \lambda_1} h \eta_{\lambda - \hat{c}} \quad (2)$$

with initial condition $\eta_\emptyset = 1$.

Theorem 1.3 ([4] The Alternating Sign Property (ASP)). *Let $n > 1$. For any partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$,*

$$\begin{aligned} \text{sign}(\eta_\lambda) &= (-1)^{|\lambda| - \lambda_1} \\ &= (-1)^{\#\text{cells under the first row of } \lambda} \end{aligned} \quad (3)$$

where $\text{sign}(\eta_\lambda)$ is 1 if η_λ is positive or -1 if η_λ is negative.

It turns out that the two terms on the right-hand side of Renteln's formula (2) can have different signs. This is the source of difficulty in the proof of the ASP by Ku and Wales which relies mainly on the recurrence. Our recurrence formula does not have this problem, thus giving a 'quicker' proof of the ASP.

To state our results, we need a new terminology. For a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, let \widehat{l}_λ denote the last row of λ and let l_λ denote the size of \widehat{l}_λ . Clearly, $l_\lambda = \lambda_r$. Also, let $\lambda - \widehat{l}_\lambda$ denote the partition obtained from λ by deleting the last row. When λ is clear from the context, we replace \widehat{l}_λ , l_λ by \widehat{l} and l respectively.

Theorem 1.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. The eigenvalues of the derangement graph Γ_n satisfy the following recurrence:*

$$\eta_\lambda = (-1)^{r-1} \lambda_r \eta_{\lambda - \widehat{l}} + (-1)^{\lambda_r} \eta_{\lambda - i} \quad (4)$$

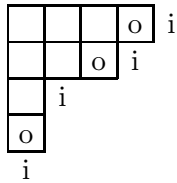
with initial condition $\eta_\emptyset = 1$.

It follows from the ASP that both of the terms on the right-hand side of (4) have the same sign.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda' = (\lambda'_1, \dots, \lambda'_r) \vdash n$. We write $\lambda <_{\text{lex}} \lambda'$, if there is a m , $1 \leq m \leq r$ such that $\lambda_i = \lambda'_i$ for all $1 \leq i \leq m-1$ and $\lambda_m < \lambda'_m$. Note that ' $<_{\text{lex}}$ ' is the usual lexicographic ordering on the partitions of n .

Let $\lambda, \lambda' \vdash n$ with λ_1 as their first part. In general, $\lambda <_{\text{lex}} \lambda'$ does not imply that $|\eta_\lambda| < |\eta_{\lambda'}|$. This has been pointed out in [4, Remark 1.4]. One of our main contributions in this paper is to show that such property holds with respect to the *dominance order*. Recall that if λ and λ' are partitions, we say that λ is *dominated* by λ' , and write $\lambda \trianglelefteq \lambda'$, if $\lambda_1 + \lambda_2 + \dots + \lambda_k \leq \lambda'_1 + \lambda'_2 + \dots + \lambda'_k$ for all positive integer k .

We give a more intuitive interpretation of the dominance order as follows. Recall that an *outside corner* of a partition λ is a box (x, y) of λ such that neither $(x+1, y)$ nor $(x, y+1)$ are boxes of λ . On the other hand, define an *inside corner* of λ as a location (x, y) which is not a box of λ , such that either $y = 1$ and $(x-1, y)$ is a box of λ , $x = 1$ and $(x, y-1)$ is a box of λ , or $(x-1, y)$ and $(x, y-1)$ are boxes of λ . For example, in the following diagram of the partition $(4, 3, 1, 1)$, the outside corners are marked with an 'o' and the inside corners with an 'i':



Let $\lambda, \lambda' \vdash n$. We write $\lambda <_1 \lambda'$, if there are m_1 and m_2 , $1 \leq m_1 < m_2 \leq r$ such that

$$\begin{aligned} \lambda &= (\lambda_1, \dots, \lambda_{m_1-1}, \lambda_{m_1}, \lambda_{m_1+1}, \dots, \lambda_{m_2-1}, \lambda_{m_2}, \lambda_{m_2+1}, \dots, \lambda_r), \\ \lambda' &= (\lambda_1, \dots, \lambda_{m_1-1}, \lambda_{m_1} + 1, \lambda_{m_1+1}, \dots, \lambda_{m_2-1}, \lambda_{m_2} - 1, \lambda_{m_2+1}, \dots, \lambda_r) \end{aligned}$$

are partitions of n . Intuitively, $\lambda <_1 \lambda'$ corresponds to sliding an outside corner of λ upwards into an inside corner of λ' .

It turns out that the dominance order can be entirely characterized in terms of the partial ordering $<_1$. We shall omit the proof of this standard result.

Lemma 1.5. *Let μ and λ be partitions of n . Then $\mu \trianglelefteq \lambda$ if and only if there exist $\mu^{(1)}, \dots, \mu^{(s)} \vdash n$ such that*

$$\mu <_1 \mu^{(1)} <_1 \dots <_1 \mu^{(s)} <_1 \lambda.$$

Using the recurrence given by Theorem 1.4, we are able to prove Theorem 1.6 and then settle affirmatively the conjecture of Ku and Wales regarding the lower and upper bounds for the absolute values of the eigenvalues of Γ_n (Theorem 1.7).

Theorem 1.6. *Let $\lambda, \lambda' \vdash n$ with λ_1 as their first part. If $\lambda \trianglelefteq \lambda'$, then*

$$|\eta_\lambda| < |\eta_{\lambda'}|.$$

Theorem 1.7 (The Ku-Wales Conjecture). *Suppose $\lambda^* \vdash n$ is the largest partition in lexicographic order among all the partitions with λ_1 as their first part. Then, for every $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$,*

$$|\eta_{(\lambda_1, 1^{n-\lambda_1})}| \leq |\eta_\lambda| \leq |\eta_{\lambda^*}|.$$

Proof. It follows from Theorem 1.6 by noting that $(\lambda_1, 1^{n-\lambda_1}) \trianglelefteq \lambda \trianglelefteq \lambda^*$, for all $\lambda \vdash n$, $\lambda \neq \lambda^*$, $(\lambda_1, 1^{n-\lambda_1})$. \square

Note that it has been shown by Ku and Wales (see [4, Theorem 1.3]) that the lower bound holds for all $\lambda_1 \geq \lfloor n/2 \rfloor$.

The paper is organized as follows. In Section 2, we introduce the shifted Schur functions developed by Okounkov and Olshanski [7] and rewrite a formula of Renteln in terms of these functions. Theorem 1.4 will then follow immediately from the property of these shifted Schur functions. Using the new recurrence formula, we provide a simpler proof of the ASP in Section 3. In Section 4, we proved Theorem 1.6, thus settling a conjecture of Ku and Wales. For the reader's convenience, in Section 5, we reproduce some the eigenvalues of the derangement graphs for small n as given in [4].

2 Shifted Schur Functions

The *Schur function* or *Schur polynomial* in n variables can be defined as the ratio of two $n \times n$ determinants

$$s_\mu(x_1, \dots, x_n) = \frac{\det [x_i^{\mu_j + n - j}]}{\det [x_i^{n - j}]}, \quad (5)$$

where μ is an arbitrary partition $\mu_1 \geq \mu_2 \geq \dots \mu_n \geq 0$ of length at most n .

An important variant of the Schur polynomial are the *shifted Schur polynomials* that was developed by Okounkov and Olshanski [7]:

$$s_\mu^*(x_1, \dots, x_n) = \frac{\det [(x_i + n - i \downarrow \mu_j + n - j)]}{\det [x_i + n - i \downarrow n - j]}, \quad (6)$$

where the symbol $(x \downarrow k)$ is the k -th *falling factorial power* of a variable x :

$$(x \downarrow k) = \begin{cases} x(x-1) \cdots (x-k+1), & \text{if } k = 1, 2, \dots \\ 1, & \text{if } k = 0. \end{cases} \quad (7)$$

Just like the ordinary Schur polynomials, the shifted Schur polynomials also satisfy the *stability property*:

$$s_\mu^*(x_1, \dots, x_n, 0) = s_\mu^*(x_1, \dots, x_n). \quad (8)$$

The stability property allow us to define the functions $s_\mu^*(x_1, x_2, \dots)$ in infinitely many variables that form a basis in the *algebra of shifted symmetric functions*, denoted by Λ^* . Every element of Λ^* may be viewed as a function $f(x_1, x_2, \dots)$ on an infinite sequence of arguments such that $x_m = 0$ for all sufficiently large m . We refer the reader to [7] for basic results on shifted symmetric functions.

For the application we have in mind, the following formula for the dimension of skew Young diagrams will be useful.

Theorem 2.1 ([7]). *Let $\mu \vdash k$ and $\lambda \vdash n$ be two partitions, where $k \leq n$ and $\mu \subseteq \lambda$. Let $\dim \lambda/\mu$ denote the number of standard tableaux of shape λ/μ ; in particular, $\dim \lambda = \dim \lambda/\emptyset$. Then*

$$\frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s_\mu^*(\lambda)}{(n \downarrow k)}, \quad (9)$$

where $s_\mu^*(\lambda) = s_\mu^*(\lambda_1, \lambda_2, \dots)$.

Theorem 2.2 ([7] Vanishing Theorem). *We have*

$$s_\mu^*(\lambda) = 0 \quad \text{unless} \quad \mu \subseteq \lambda, \quad (10)$$

$$s_\mu^*(\mu) = H(\mu), \quad (11)$$

where $H(\mu) = \prod_{\alpha \in \mu} h(\alpha)$ is the product of the hook lengths of all boxes of μ .

As an example of shifted symmetric functions, set $h_k^* = s_{(k)}^*$ where (k) is the partition of k whose Young diagram consists of just one row. These are called the *complete shifted symmetric functions*. They are shifted analogues of the complete homogeneous symmetric functions. We shall require the following properties of h_k^* .

Proposition 2.3 ([7]). *The complete shifted symmetric functions h_k^* can be written as*

$$h_k^*(x_1, x_2, \dots) = \sum_{1 \leq i_1 \leq \dots \leq i_k < \infty} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots x_{i_k}. \quad (12)$$

Corollary 2.4. *The complete shifted symmetric functions h_k^* satisfy the following recurrence:*

$$h_k^*(x_1, \dots, x_n) = x_n h_{k-1}^*(x_1 - 1, \dots, x_n - 1) + h_k^*(x_1, \dots, x_{n-1}). \quad (13)$$

Proof. In view of the stability property and Proposition 2.3, we have

$$h_k^*(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots x_{i_k}.$$

Therefore,

$$\begin{aligned} h_k^*(x_1, \dots, x_n) &= x_n \left(\sum_{1 \leq i_1 \leq \dots \leq i_{k-1} \leq n} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots (x_{i_{k-1}} - 1) \right) \\ &\quad + \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n-1} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots x_{i_k} \\ &= x_n h_{k-1}^*(x_1 - 1, \dots, x_n - 1) + h_k^*(x_1, \dots, x_{n-1}). \end{aligned}$$

□

Recall the following formula due to Renteln [8, Theorem 3.2].

Theorem 2.5 ([8]). *The eigenvalues of the derangement graph Γ_n are given by*

$$\eta_\lambda = \sum_{k=0}^n (-1)^{n-k} (n \downarrow k) \frac{\dim \lambda / (k)}{\dim \lambda} \quad (14)$$

Therefore, it follows immediately from Theorem 2.1 and Theorem 2.5 that

Corollary 2.6. *The eigenvalues of the derangement graph Γ_n are given by*

$$\begin{aligned} \eta_\lambda &= \sum_{k=0}^n (-1)^{n-k} s_{(k)}^*(\lambda) \\ &= \sum_{k=0}^n (-1)^{n-k} h_k^*(\lambda). \end{aligned} \quad (15)$$

Proof of Theorem 1.4.

Set $\eta'_\lambda = \sum_{k=0}^n (-1)^k h_k^*(\lambda)$. By the Vanishing Theorem (Theorem 2.2) and Corollary 2.6, we can write

$$\eta'_\lambda = \sum_{k=0}^{\infty} (-1)^k h_k^*(\lambda)$$

so that

$$\eta'_\lambda = (-1)^{|\lambda|} \eta_\lambda.$$

By (13),

$$\begin{aligned} \eta'_\lambda &= \sum_{k=0}^{\infty} \left((-1)^k (\lambda_r h_{k-1}^*(\lambda_1 - 1, \dots, \lambda_r - 1) + h_k^*(\lambda_1, \dots, \lambda_{r-1})) \right) \\ &= -\lambda_r \sum_{k=0}^{\infty} (-1)^{k-1} h_{k-1}^*(\lambda_1 - 1, \dots, \lambda_r - 1) + \sum_{k=0}^{\infty} (-1)^k h_k^*(\lambda_1, \dots, \lambda_{r-1}) \\ &= -\lambda_r \eta'_{\lambda - \hat{e}} + \eta'_{\lambda - \hat{i}} \\ &= -\lambda_r (-1)^{|\lambda - \hat{e}|} \eta_{\lambda - \hat{e}} + (-1)^{|\lambda - \hat{i}|} \eta_{\lambda - \hat{i}} \\ (-1)^{|\lambda|} \eta_\lambda &= \lambda_r (-1)^{1+|\lambda|-r} \eta_{\lambda - \hat{e}} + (-1)^{|\lambda|-\lambda_r} \eta_{\lambda - \hat{i}} \\ \eta_\lambda &= (-1)^{r-1} \lambda_r \eta_{\lambda - \hat{e}} + (-1)^{\lambda_r} \eta_{\lambda - \hat{i}}. \end{aligned}$$

□

3 A simpler proof of the Alternating Sign Property

We prove by induction on $|\lambda|$. Obviously, the property holds for all small partitions. By the inductive hypothesis,

$$\begin{aligned} \text{sign}((-1)^{r-1} \eta_{\lambda - \hat{e}}) &= (-1)^{r-1} (-1)^{|\lambda - \hat{e}| - (\lambda_1 - 1)} \\ &= (-1)^{r-1+|\lambda|-r-\lambda_1+1} \\ &= (-1)^{|\lambda|-\lambda_1}. \end{aligned}$$

Similarly,

$$\begin{aligned}
\text{sign} \left((-1)^{\lambda_r} \eta_{\lambda-\hat{i}} \right) &= (-1)^{\lambda_r} (-1)^{|\lambda-\hat{i}|-\lambda_1} \\
&= (-1)^{\lambda_r+|\lambda|-\lambda_r-\lambda_1} \\
&= (-1)^{|\lambda|-\lambda_1}.
\end{aligned}$$

By the recurrence formula (4), we deduce that

$$\text{sign}(\eta_\lambda) = (-1)^{|\lambda|-\lambda_1}.$$

□

4 Some preliminary lemmas

For convenience, let us write

$$f(\lambda_1, \lambda_2, \dots, \lambda_r) = |\eta_{(\lambda_1, \lambda_2, \dots, \lambda_r)}|.$$

Then by Theorem 1.3 and Theorem 1.4, we have

$$f(\lambda_1, \lambda_2, \dots, \lambda_r) = \lambda_r f(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_r - 1) + f(\lambda_1, \lambda_2, \dots, \lambda_{r-1}). \quad (16)$$

By abuse of notation, in this section we shall use the symbol λ to denote a positive integer instead of a partition.

Lemma 4.1.

$$\begin{aligned}
h_0^*(\lambda) &= 1, \\
h_1^*(\lambda) &= \lambda, \\
h_k^*(\lambda) &= (\lambda - k + 1)(\lambda - k + 2) \cdots (\lambda - 1)(\lambda), \quad \text{for } k \geq 2.
\end{aligned}$$

Proof. It follows easily from Proposition 2.3. □

Lemma 4.2. *For any $1 < m \leq r$,*

$$f(\lambda_1, \lambda_2, \dots, \lambda_r) = \sum_{k=0}^{\lambda_m} h_k^*(\lambda_m, \dots, \lambda_r) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{m-1} - k).$$

Proof. Repeatedly applying equation (16) and by Lemma 4.1, we obtain

$$\begin{aligned}
& f(\lambda_1, \lambda_2, \dots, \lambda_r) \\
&= h_1^*(\lambda_r) f(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_r - 1) + h_0^*(\lambda_r) f(\lambda_1, \lambda_2, \dots, \lambda_{r-1}) \\
&= (\lambda_r)(\lambda_r - 1) f(\lambda_1 - 2, \lambda_2 - 2, \dots, \lambda_r - 2) + \sum_{k=0}^1 h_k^*(\lambda_r) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{r-1} - k) \\
&= (\lambda_r)(\lambda_r - 1)(\lambda_r - 2) f(\lambda_1 - 3, \lambda_2 - 3, \dots, \lambda_r - 3) + \sum_{k=0}^2 h_k^*(\lambda_r) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{r-1} - k) \\
&\quad \vdots \\
&= \sum_{k=0}^{\lambda_r} h_k^*(\lambda_r) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{r-1} - k). \tag{17}
\end{aligned}$$

Thus the lemma holds for $m = r$. Assume that it holds for some m_0 , $2 < m_0 \leq r$. We shall show that it also holds for $m_0 - 1$.

By assumption, the following equation holds:

$$f(\lambda_1, \lambda_2, \dots, \lambda_r) = \sum_{k=0}^{\lambda_{m_0}} h_k^*(\lambda_{m_0}, \dots, \lambda_r) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{m_0-1} - k). \tag{18}$$

By applying equation (17),

$$\begin{aligned}
& f(\lambda_1, \lambda_2, \dots, \lambda_r) \\
&= \sum_{k=0}^{\lambda_{m_0}} h_k^*(\lambda_{m_0}, \dots, \lambda_r) \left(\sum_{j=0}^{\lambda_{m_0-1}-k} h_j^*(\lambda_{m_0-1} - k) f(\lambda_1 - k - j, \lambda_2 - k - j, \dots, \lambda_{m_0-2} - k - j) \right) \\
&= \sum_{k=0}^{\lambda_{m_0}} \sum_{j=0}^{\lambda_{m_0-1}-k} h_k^*(\lambda_{m_0}, \dots, \lambda_r) h_j^*(\lambda_{m_0-1} - k) f(\lambda_1 - k - j, \lambda_2 - k - j, \dots, \lambda_{m_0-2} - k - j). \tag{19}
\end{aligned}$$

Now by collecting all the terms with $k + j = j_0$, equation (19) becomes

$$\begin{aligned}
& f(\lambda_1, \lambda_2, \dots, \lambda_r) \\
&= \sum_{j_0=0}^{\lambda_{m_0-1}} \left(\sum_{\substack{k+j=j_0, \\ 0 \leq k \leq \lambda_{m_0}}} h_j^*(\lambda_{m_0-1} - k) h_k^*(\lambda_{m_0}, \dots, \lambda_r) \right) f(\lambda_1 - j_0, \lambda_2 - j_0, \dots, \lambda_{m_0-2} - j_0). \tag{20}
\end{aligned}$$

By Proposition 2.3,

$$h_{j_0}^*(\lambda_{m_0-1}, \lambda_{m_0}, \dots, \lambda_r) = \sum_{\substack{k+j=j_0, \\ 0 \leq k \leq \lambda_{m_0}}} h_j^*(\lambda_{m_0-1} - k) h_k^*(\lambda_{m_0}, \dots, \lambda_r).$$

Thus, by induction the lemma follows. \square

Lemma 4.3.

$$\begin{aligned} h_0^*(\lambda_1, \lambda_2, \dots, \lambda_r) &= 1, \\ h_1^*(\lambda_1, \lambda_2, \dots, \lambda_r) &= \lambda_1 + \lambda_2 + \dots + \lambda_r. \end{aligned}$$

Proof. It follows easily from Proposition 2.3. □

Lemma 4.4. *If $\lambda_s \leq \lambda$ and $2 \leq k \leq \lambda$, then*

$$h_k^*(\lambda, \lambda_s) < h_k^*(\lambda + 1, \lambda_s - 1).$$

Proof. By Proposition 2.3,

$$h_k^*(x, y) = \sum_{j=0}^k (x - j \downarrow k - j)(y \downarrow j).$$

Therefore,

$$h_k^*(\lambda + 1, \lambda_s - 1) - h_k^*(\lambda, \lambda_s - 1) = \sum_{j=0}^{k-1} (k - j)(\lambda - j \downarrow k - j - 1)(\lambda_s - 1 \downarrow j), \quad (21)$$

and

$$\begin{aligned} h_k^*(\lambda, \lambda_s) - h_k^*(\lambda, \lambda_s - 1) &= \sum_{j=1}^k j(\lambda - j \downarrow k - j)(\lambda_s - 1 \downarrow j - 1). \\ &= \sum_{j=0}^{k-1} (j + 1)(\lambda - j - 1 \downarrow k - j - 1)(\lambda_s - 1 \downarrow j). \end{aligned} \quad (22)$$

We shall compare equation (21) with equation (22). For $0 \leq j < \frac{k-1}{2}$, the j -th and $(k-1-j)$ -th term of the right side of equation (21) are

$$(k - j)(\lambda - j \downarrow k - j - 1)(\lambda_s - 1 \downarrow j), \quad (23)$$

$$(j + 1)(\lambda - k + 1 + j \downarrow j)(\lambda_s - 1 \downarrow k - 1 - j). \quad (24)$$

On the other hand, the j -th and $(k-1-j)$ -th term of the right side of equation (22) are

$$(j + 1)(\lambda - j - 1 \downarrow k - j - 1)(\lambda_s - 1 \downarrow j), \quad (25)$$

$$(k - j)(\lambda - k + j \downarrow j)(\lambda_s - 1 \downarrow k - 1 - j). \quad (26)$$

When $j = 0$, the sum (23) + (24) - (25) - (26) is

$$\begin{aligned} &k(\lambda \downarrow k - 1) + (\lambda_s - 1 \downarrow k - 1) - (\lambda - 1 \downarrow k - 1) - k(\lambda_s - 1 \downarrow k - 1) \\ &= ((\lambda \downarrow k - 1) - (\lambda - 1 \downarrow k - 1)) + (k - 1)((\lambda \downarrow k - 1) - (\lambda_s - 1 \downarrow k - 1)) \\ &= ((k - 1)(\lambda - 1 \downarrow k - 2)) + (k - 1)((\lambda \downarrow k - 1) - (\lambda_s - 1 \downarrow k - 1)) \\ &> 0, \end{aligned} \quad (27)$$

where the last inequality follows from $k \geq 2$ and $\lambda \geq \lambda_s > \lambda_s - 1$.

Now for $1 \leq j < \frac{k-1}{2}$, (23) – (25) is

$$((k-j)(\lambda-j) - (j+1)(\lambda-k+1))(\lambda-j-1 \downarrow k-j-2)(\lambda_s-1 \downarrow j), \quad (28)$$

and (24) – (26) is

$$((j+1)(\lambda-k+1+j) - (k-j)(\lambda-k+1))(\lambda-k+j \downarrow j-1)(\lambda_s-1 \downarrow k-1-j). \quad (29)$$

Since $\lambda \geq \lambda_s$ and $j < \frac{k-1}{2}$,

$$\begin{aligned} (k-j)(\lambda-j) - (j+1)(\lambda-k+1) &= (k-(2j+1))\lambda + (k-1) + j(j-1) > 0, \\ (\lambda-j-1 \downarrow k-j-2)(\lambda_s-1 \downarrow j) &\geq (\lambda-k+j \downarrow j-1)(\lambda_s-1 \downarrow k-1-j). \end{aligned}$$

Therefore the sum (28)+(29) is at least

$$((k-j)(k-j-1) + (j+1)j)(\lambda-k+j \downarrow j-1)(\lambda_s-1 \downarrow k-1-j) > 0. \quad (30)$$

If k is odd, then j can take value $\frac{k-1}{2}$. The $\frac{k-1}{2}$ -th term on the right side of (21) is

$$\frac{k+1}{2} \left(\lambda - \frac{k-1}{2} \downarrow \frac{k-1}{2} \right) \left(\lambda_s - 1 \downarrow \frac{k-1}{2} \right), \quad (31)$$

and the $\frac{k-1}{2}$ -th term on the right side of (22) is

$$\frac{k+1}{2} \left(\lambda - \frac{k+1}{2} \downarrow \frac{k-1}{2} \right) \left(\lambda_s - 1 \downarrow \frac{k-1}{2} \right). \quad (32)$$

Note that (31) – (32) is

$$\frac{k+1}{2} \frac{k-1}{2} \left(\lambda - \frac{k+1}{2} \downarrow \frac{k-1}{2} - 1 \right) \left(\lambda_s - 1 \downarrow \frac{k-1}{2} \right) > 0. \quad (33)$$

From equations (27), (30) and (33), we deduce that

$$\begin{aligned} &h_k^*(\lambda+1, \lambda_s-1) - h_k^*(\lambda, \lambda_s) \\ &= (h_k^*(\lambda+1, \lambda_s-1) - h_k^*(\lambda, \lambda_s-1)) \\ &\quad - (h_k^*(\lambda, \lambda_s) - h_k^*(\lambda, \lambda_s-1)) \\ &> 0. \end{aligned}$$

□

Lemma 4.5. *Let $l \geq 1$ and*

$$h_k^*(\lambda, \lambda^l, \lambda) = h_k^*(\lambda, \underbrace{\lambda, \dots, \lambda}_{l \text{ times}}, \lambda).$$

If $2 \leq k \leq \lambda$, then

$$h_k^*(\lambda, \lambda^l, \lambda) < h_k^*(\lambda+1, \lambda^l, \lambda-1).$$

Proof. By Proposition 2.3,

$$\begin{aligned} & (x - j - r \downarrow k - j - r)(\lambda - j \downarrow r)(y \downarrow j) \\ &= (x - k + 1) \cdots (x - j - r)(\lambda - j - r + 1) \cdots (\lambda - j)(y - j + 1) \cdots (y), \end{aligned}$$

is a term in the sum of $h_k^*(x, \lambda^l, y)$. In fact, there are $\binom{r+l-1}{l-1}$ such terms. Therefore

$$h_k^*(x, \lambda^l, y) = \sum_{j=0}^k \sum_{r=0}^{k-j} \binom{r+l-1}{l-1} (x - j - r \downarrow k - j - r)(\lambda - j \downarrow r)(y \downarrow j). \quad (34)$$

From (34),

$$\begin{aligned} & h_k^*(x+1, \lambda^l, y) - h_k^*(x, \lambda^l, y) \\ &= \sum_{j=0}^{k-1} \sum_{r=0}^{k-1-j} (k - j - r) \binom{r+l-1}{l-1} (x - j - r \downarrow k - 1 - j - r)(\lambda - j \downarrow r)(y \downarrow j). \end{aligned} \quad (35)$$

Now replacing x with λ and y with $\lambda - 1$ in (35), we obtain

$$\begin{aligned} & h_k^*(\lambda+1, \lambda^l, \lambda-1) - h_k^*(\lambda, \lambda^l, \lambda-1) \\ &= \sum_{j=0}^{k-1} \sum_{r=0}^{k-1-j} (k - j - r) \binom{r+l-1}{l-1} (\lambda - j \downarrow k - 1 - j)(\lambda - 1 \downarrow j) \\ &= \sum_{j=0}^{k-1} \binom{k-j+l}{l+1} (\lambda - j \downarrow k - 1 - j)(\lambda - 1 \downarrow j) \\ &= \sum_{j=0}^{k-1} \binom{k-j+l}{l+1} (\lambda - j)(\lambda - 1 \downarrow k - 2) \\ &> \sum_{j=0}^{k-1} \binom{k-j+l}{l+1} (\lambda - k + 1)(\lambda - 1 \downarrow k - 2) \\ &= \sum_{j=0}^{k-1} \binom{k-j+l}{l+1} (\lambda - 1 \downarrow k - 1) \\ &= \binom{k+l+1}{l+2} (\lambda - 1 \downarrow k - 1). \end{aligned} \quad (36)$$

From (34),

$$\begin{aligned} & h_k^*(x, \lambda^l, y) - h_k^*(x, \lambda^l, y-1) \\ &= \sum_{j=1}^k \sum_{r=0}^{k-j} j \binom{r+l-1}{l-1} (x - j - r \downarrow k - j - r)(\lambda - j \downarrow r)(y - 1 \downarrow j - 1). \end{aligned} \quad (37)$$

Now replacing x with λ and y with λ in (37), we obtain

$$\begin{aligned}
& h_k^*(\lambda, \lambda^l, \lambda) - h_k^*(\lambda, \lambda^l, \lambda - 1) \\
&= \sum_{j=1}^k \sum_{r=0}^{k-j} j \binom{r+l-1}{l-1} (\lambda - j \downarrow k - j) (\lambda - 1 \downarrow j - 1) \\
&= \sum_{j=1}^k \sum_{r=0}^{k-j} j \binom{r+l-1}{l-1} (\lambda - 1 \downarrow k - 1) \\
&= \sum_{j=1}^k j \binom{k-j+l}{l} (\lambda - 1 \downarrow k - 1) \\
&= \binom{k+l+1}{l+2} (\lambda - 1 \downarrow k - 1).
\end{aligned} \tag{38}$$

By equations (36) and (38), we deduce that

$$h_k^*(\lambda + 1, \lambda^l, \lambda - 1) - h_k^*(\lambda, \lambda^l, \lambda) > 0.$$

□

Lemma 4.6. *Let $r \geq 3$. If*

$$\begin{aligned}
& (\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1}, \lambda_r), \\
& (\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1} + 1, \lambda_r - 1),
\end{aligned}$$

are two partitions of n , then

$$f(\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1}, \lambda_r) < f(\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1} + 1, \lambda_r - 1).$$

Proof. By Lemma 4.2,

$$f(\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1}, \lambda_r) = \sum_{k=0}^{\lambda_{r-1}} h_k^*(\lambda_{r-1}, \lambda_r) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{r-2} - k),$$

and

$$\begin{aligned}
f(\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1} + 1, \lambda_r - 1) &= \sum_{k=0}^{\lambda_{r-1}+1} h_k^*(\lambda_{r-1} + 1, \lambda_r - 1) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{r-2} - k) \\
&\geq \sum_{k=0}^{\lambda_{r-1}} h_k^*(\lambda_{r-1} + 1, \lambda_r - 1) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_{r-2} - k).
\end{aligned}$$

The lemma then follows from Lemma 4.3 and Lemma 4.4. □

Lemma 4.7. *If $l \geq 1$ and*

$$\begin{aligned}
& (\lambda_1, \dots, \lambda_r, \lambda, \lambda^l, \lambda), \\
& (\lambda_1, \dots, \lambda_r, \lambda + 1, \lambda^l, \lambda - 1),
\end{aligned}$$

are two partitions of n , then

$$f(\lambda_1, \dots, \lambda_r, \lambda, \lambda^l, \lambda) < f(\lambda_1, \dots, \lambda_r, \lambda + 1, \lambda^l, \lambda - 1).$$

Proof. By Lemma 4.2,

$$f(\lambda_1, \dots, \lambda_r, \lambda, \lambda^l, \lambda) = \sum_{k=0}^{\lambda} h_k^*(\lambda, \lambda^l, \lambda) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_r - k),$$

and

$$\begin{aligned} f(\lambda_1, \dots, \lambda_r, \lambda + 1, \lambda^l, \lambda - 1) &= \sum_{k=0}^{\lambda+1} h_k^*(\lambda + 1, \lambda^l, \lambda - 1) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_r - k) \\ &\geq \sum_{k=0}^{\lambda} h_k^*(\lambda + 1, \lambda^l, \lambda - 1) f(\lambda_1 - k, \lambda_2 - k, \dots, \lambda_r - k). \end{aligned}$$

The lemma then follows from Lemma 4.3 and Lemma 4.5. \square

5 Proof of Theorem 1.6

Proof. By Lemma 1.5, it is sufficient to show that the inequality holds for $\lambda <_1 \lambda'$, i.e. if $\lambda <_1 \lambda'$ then $|\eta_\lambda| < |\eta_{\lambda'}|$.

Let $2 \leq m_1 < m_2 \leq r$ be such that

$$\begin{aligned} \lambda &= (\lambda_1, \dots, \lambda_{m_1-1}, \lambda_{m_1}, \lambda_{m_1+1}, \dots, \lambda_{m_2-1}, \lambda_{m_2}, \lambda_{m_2+1}, \dots, \lambda_r) \\ \lambda' &= (\lambda_1, \dots, \lambda_{m_1-1}, \lambda_{m_1} + 1, \lambda_{m_1+1}, \dots, \lambda_{m_2-1}, \lambda_{m_2} - 1, \lambda_{m_2+1}, \dots, \lambda_r). \end{aligned}$$

We shall prove by induction on n . Clearly, Theorem 1.6 holds for small values of n . We shall distinguish two cases.

Case 1. $m_2 \neq r$. Then by Theorem 1.3 and Theorem 1.4,

$$|\eta_\lambda| = \lambda_r |\eta_{\lambda-\hat{c}}| + |\eta_{\lambda-\hat{l}}|.$$

Note that $\lambda - \hat{c} <_1 \lambda' - \hat{c}$ and $\lambda - \hat{l} <_1 \lambda' - \hat{l}$. So, by induction,

$$|\eta_\lambda| = \lambda_r |\eta_{\lambda-\hat{c}}| + |\eta_{\lambda-\hat{l}}| < \lambda_r |\eta_{\lambda'-\hat{c}}| + |\eta_{\lambda'-\hat{l}}| = |\eta_{\lambda'}|.$$

Case 2. $m_2 = r$. If $m_1 = r - 1$, then it follows from Lemma 4.6 that $|\eta_\lambda| < |\eta_{\lambda'}|$. Suppose $m_1 < r - 1$.

Let m_3 be the largest integer such that

$$\lambda'' = (\lambda_1, \dots, \lambda_{m_3-1}, \lambda_{m_3} + 1, \lambda_{m_3+1}, \dots, \lambda_r - 1),$$

is a partition of n . Note that $m_1 \leq m_3$. By the choice of m_3 , we must have

$$\lambda_{m_3-1} > \lambda_{m_3} = \lambda_{m_3+1} = \dots = \lambda_{r-1}.$$

If $\lambda_r = \lambda_{r-1}$, then by Lemma 4.7, $|\eta_\lambda| < |\eta_{\lambda''}|$. If $\lambda_r < \lambda_{r-1}$, then by Case 1, $|\eta_\lambda| < |\eta_{\lambda''}|$, where

$$\lambda''' = (\lambda_1, \dots, \lambda_{m_3-1}, \lambda_{m_3} + 1, \lambda_{m_3+1}, \dots, \lambda_{r-1} - 1, \lambda_r).$$

By Lemma 4.6, $|\eta_{\lambda''}| < |\eta_{\lambda'''}|$. Thus $|\eta_\lambda| < |\eta_{\lambda''}|$.

In either case, $|\eta_\lambda| < |\eta_{\lambda''}|$. If $m_1 = m_3$, then we are done. If $m_1 < m_3$, then by Case 1, $|\eta_{\lambda''}| < |\eta_{\lambda'}|$. Hence $|\eta_\lambda| < |\eta_{\lambda'}|$. This completes the proof of the theorem. \square

6 Some Values of η_λ

In this section we reproduce some of the eigenvalues of Γ_n for small n as given in [4].

$$n = 2$$

λ	η_λ
2	1
1^2	-1

$$n = 3$$

λ	η_λ
3	2
$2, 1$	-1
1^3	2

$$n = 4$$

λ	η_λ		λ	η_λ
4	9		$2, 1^2$	1
$3, 1$	-3		1^4	-3
$2, 2$	3			

$$n = 5$$

λ	η_λ		λ	η_λ
5	44		$2^2, 1$	-4
$4, 1$	-11		$2, 1^3$	-1
$3, 2$	4		1^5	4
$3, 1^2$	4			

$$n = 6$$

λ	η_λ		λ	η_λ
6	265		$3, 1^3$	-5
$5, 1$	-53		2^3	7
$4, 2$	15		$2^2, 1^2$	5
$4, 1^2$	13		$2, 1^4$	1
3^2	-11		1^6	-5
$3, 2, 1$	-5			

$$n = 7$$

λ	η_λ		λ	η_λ
7	1854		$3, 2^2$	6
$6, 1$	-309		$3, 2, 1^2$	6
$5, 2$	66		$3, 1^4$	6
$5, 1, 1$	62		$2^3, 1$	-9
$4, 3$	-21		$2^2, 1^3$	-6
$4, 2, 1$	-18		$2, 1^5$	-1
$4, 1^3$	-15		1^7	6
$3^2, 1$	14			

$$n = 8$$

λ	η_λ		λ	η_λ
8	14833		$4, 1^4$	17
$7, 1$	-2119		$3^2, 2$	-19
$6, 2$	371		$3^2, 1^2$	-17
$6, 1^2$	353		$3, 2^2, 1$	-7
$5, 3$	-89		$3, 2, 1^3$	-7
$5, 2, 1$	-77		$3, 1^5$	-7
$5, 1^3$	-71		2^4	13
4^2	53		$2^3, 1^2$	11
$4, 3, 1$	25		$2^2, 1^4$	7
$4, 2^2$	23		$2, 1^6$	1
$4, 2, 1^2$	21		1^8	-7

$$n = 9$$

λ	η_λ		λ	η_λ
9	133496		$4, 2^2, 1$	-27
$8, 1$	-16687		$4, 2, 1^3$	-24
$7, 2$	2472		$4, 1^5$	-19
$7, 1^2$	2384		3^3	32
$6, 3$	-463		$3^2, 2, 1$	23
$6, 2, 1$	-424		$3^2, 1^3$	20
$6, 1^3$	-397		$3, 2^3$	8
$5, 4$	128		$3, 2^2, 1^2$	8
$5, 3, 1$	104		$3, 2, 1^4$	8
$5, 2^2$	92		$3, 1^6$	8
$5, 2, 1^2$	88		$2^4, 1$	-16
$5, 1^4$	80		$2^3, 1^3$	-13
$4^2, 1$	-64		$2^2, 1^5$	-8
$4, 3, 2$	-31		$2, 1^7$	-1
$4, 3, 1^2$	-29		1^9	8

$n = 10$

λ	η_λ		λ	η_λ		λ	η_λ		λ	η_λ
10	1334961		$6, 1^4$	441		$4, 3, 2, 1$	36		$3, 2^2, 1^3$	-9
9, 1	-148329		5, 5	-309		$4, 3, 1^3$	33		$3, 2, 1^5$	-9
8, 2	19071		$5, 4, 1$	-149		$4, 2^3$	33		$3, 1^7$	-9
$8, 1^2$	18541		$5, 3, 2$	-125		$4, 2^2, 1^2$	31		2^5	21
7, 3	-2967		$5, 3, 1^2$	-119		$4, 2, 1^4$	27		$2^4, 1^2$	19
$7, 2, 1$	-2781		$5, 2^2, 1$	-105		$4, 1^6$	21		$2^3, 1^4$	15
$7, 1^3$	-2649		$5, 2, 1^3$	-99		$3^3, 1$	-39		$2^2, 1^6$	9
6, 4	621		$5, 1^5$	-89		$3^2, 2^2$	-29		$2, 1^8$	1
$6, 3, 1$	529		$4^2, 2$	81		$3^2, 2, 1^2$	-27		1^{10}	-9
$6, 2^2$	495		$4^2, 1^2$	75		$3^2, 1^4$	-23			
$6, 2, 1^2$	477		$4, 3^2$	39		$3, 2^3, 1$	-9			

$n = 11, \lambda_1 \geq 5$

λ	η_λ		λ	η_λ		λ	η_λ		λ	η_λ
11	14684570		$7, 3, 1$	3338		$6, 2^2, 1$	-557		$5, 3, 1^3$	134
10, 1	-1468457		$7, 2^2$	3178		$6, 2, 1^3$	-530		$5, 2^3$	122
9, 2	166870		$7, 2, 1^2$	3090		$6, 1^5$	-485		$5, 2^2, 1^2$	118
$9, 1^2$	163162		$7, 1^4$	2914		$5^2, 1$	362		$5, 2, 1^4$	110
8, 3	-22249		6, 5	-905		$5, 4, 2$	178		$5, 1^6$	98
$8, 2, 1$	-21190		$6, 4, 1$	-710		$5, 4, 1^2$	170			
$8, 1^3$	-20395		$6, 3, 2$	-617		$5, 3^2$	158			
7, 4	3706		$6, 3, 1^2$	-595		$5, 3, 2, 1$	143			

$n = 12, \lambda_1 \geq 6$

λ	η_λ		λ	η_λ		λ	η_λ		λ	η_λ
12	176214841		$8, 3, 1$	24721		$7, 2^2, 1$	-3531		$6, 3, 2, 1$	694
11, 1	-16019531		$8, 2^2$	23839		$7, 2, 1^3$	-3399		$6, 3, 1^3$	661
10, 2	1631619		$8, 2, 1^2$	23309		$7, 1^5$	-3179		$6, 2^3$	637
$10, 1^2$	1601953		$8, 1^4$	22249		6^2	2119		$6, 2^2, 1^2$	619
9, 3	-190709		7, 5	-4959		$6, 5, 1$	1033		$6, 2, 1^4$	583
$9, 2, 1$	-183557		$7, 4, 1$	-4169		$6, 4, 2$	829		$6, 1^6$	529
$9, 1^3$	-177995		$7, 3, 2$	-3815		$6, 4, 1^2$	799			
8, 4	26701		$7, 3, 1^2$	-3709		$6, 3^2$	739			

$n = 13, \lambda_1 \geq 6$

λ	η_λ		λ	η_λ		λ	η_λ		λ	η_λ
13	2290792932		$9, 1^4$	192828		$7, 4, 1^2$	4632		$6, 4, 3$	-996
12, 1	-190899411		8, 5	-33363		$7, 3^2$	4452		$6, 4, 2, 1$	-933
11, 2	17621484		$8, 4, 1$	-29668		$7, 3, 2, 1$	4239		$6, 4, 1^3$	-888
$11, 1^2$	17354492		$8, 3, 2$	-27811		$7, 3, 1^3$	4080		$6, 3^2, 1$	-831
10, 3	-1835571		$8, 3, 1^2$	-27193		$7, 2^3$	3972		$6, 3, 2^2$	-793
$10, 2, 1$	-1779948		$8, 2^2, 1$	-26223		$7, 2^2, 1^2$	3884		$6, 3, 2, 1^2$	-771
$10, 1^3$	-1735449		$8, 2, 1^3$	-25428		$7, 2, 1^4$	3708		$6, 3, 1^4$	-727
9, 4	222492		$8, 1^5$	-24103		$7, 1^6$	3444		$6, 2^3, 1$	-708
$9, 3, 1$	209780		7, 6	7284		$6^2, 1$	-2428		$6, 2^2, 1^3$	-681
$9, 2^2$	203952		$7, 5, 1$	5580		$6, 5, 2$	-1203		$6, 2, 1^5$	-636
$9, 2, 1^2$	200244		$7, 4, 2$	4764		$6, 5, 1^2$	-1161		$6, 1^7$	-573

$n = 15$

λ	η_λ	λ	η_λ	λ	η_λ	λ	η_λ
15	481066515734	$7^2, 1$	18806	$6, 2, 1^7$	-742	$4, 3^2, 2^2, 1$	-77
14, 1	-34361893981	$7, 6, 2$	9350	$6, 1^9$	-661	$4, 3^2, 2, 1^3$	-74
13, 2	2672591754	$7, 6, 1^2$	9094	5^3	1214	$4, 3^2, 1^5$	-69
$13, 1^2$	2643222614	$7, 5, 3$	7446	$5^2, 4, 1$	859	$4, 3, 2^4$	-73
12, 3	-229079293	$7, 5, 2, 1$	7089	$5^2, 3, 2$	742	$4, 3, 2^3, 1^2$	-71
$12, 2, 1$	-224273434	$7, 5, 1^3$	6822	$5^2, 3, 1^2$	714	$4, 3, 2^3, 1^4$	-67
$12, 1^3$	-220268551	$7, 4^2$	6662	$5^2, 2^2, 1$	662	$4, 3, 2, 1^6$	-61
11, 4	22026854	$7, 4, 3, 1$	6174	$5^2, 2, 1^3$	629	$4, 3, 1^8$	-53
$11, 3, 1$	21211046	$7, 4, 2^2$	5954	$5^2, 1^5$	574	$4, 2^5, 1$	-66
$11, 2^2$	20825390	$7, 4, 2, 1^2$	5822	$5, 4^2, 2$	374	$4, 2^4, 1^3$	-63
$11, 2, 1^2$	20558398	$7, 4, 1^4$	5558	$5, 4^2, 1^2$	362	$4, 2^3, 1^5$	-58
$11, 1^4$	20024414	$7, 3^2, 2$	5566	$5, 4, 3^2$	350	$4, 2^2, 1^7$	-51
10, 5	-2447421	$7, 3^2, 1^2$	5442	$5, 4, 3, 2, 1$	329	$4, 2, 1^9$	-42
$10, 4, 1$	-2288506	$7, 3, 2^2, 1$	5246	$5, 4, 3, 1^3$	314	$4, 1^{11}$	-31
$10, 3, 2$	-2202685	$7, 3, 2, 1^3$	5087	$5, 4, 2^3$	302	3^5	134
$10, 3, 1^2$	-2169311	$7, 3, 1^5$	4822	$5, 4, 2^2, 1^2$	294	$3^4, 2, 1$	119
$10, 2^2, 1$	-2121105	$7, 2^4$	4854	$5, 4, 2, 1^4$	278	$3^4, 1^3$	110
$10, 2, 1^3$	-2076606	$7, 2^3, 1^2$	4766	$5, 4, 1^6$	254	$3^3, 2^3$	98
$10, 1^5$	-2002441	$7, 2^2, 1^4$	4590	$5, 3^3, 1$	290	$3^3, 2^2, 1^2$	94
9, 6	333674	$7, 2, 1^6$	4326	$5, 3^2, 2^2$	274	$3^3, 2, 1^4$	86
$9, 5, 1$	293702	$7, 1^8$	3974	$5, 3^2, 2, 1^2$	266	$3^3, 1^6$	74
$9, 4, 2$	271934	$6^2, 3$	-3430	$5, 3^2, 1^4$	250	$3^2, 2^4, 1$	62
$9, 4, 1^2$	266990	$6^2, 2, 1$	-3205	$5, 3, 2^3, 1$	239	$3^2, 2^3, 1^3$	59
$9, 3^2$	262226	$6^2, 1^3$	-3046	$5, 3, 2^2, 1^3$	230	$3^2, 2^2, 1^5$	54
$9, 3, 2, 1$	254279	$6, 5, 4$	-1789	$5, 3, 2, 1^5$	215	$3^2, 2, 1^7$	47
$9, 3, 1^3$	247922	$6, 5, 3, 1$	-1617	$5, 3, 1^7$	194	$3^2, 1^9$	38
$9, 2^3$	244742	$6, 5, 2^2$	-1543	$5, 2^5$	194	$3, 2^6$	14
$9, 2^2, 1^2$	241034	$6, 5, 2, 1^2$	-1501	$5, 2^4, 1^2$	190	$3, 2^5, 1^2$	14
$9, 2, 1^4$	233618	$6, 5, 1^4$	-1417	$5, 2^3, 1^4$	182	$3, 2^4, 1^4$	14
$9, 1^6$	222494	$6, 4^2, 1$	-1411	$5, 2^2, 1^6$	170	$3, 2^3, 1^6$	14
8, 7	-65821	$6, 4, 3, 2$	-1282	$5, 2, 1^8$	154	$3, 2^2, 1^8$	14
$8, 6, 1$	-49546	$6, 4, 3, 1^2$	-1246	$5, 1^{10}$	134	$3, 2, 1^{10}$	14
$8, 5, 2$	-41701	$6, 4, 2^2, 1$	-1181	$4^3, 3$	-331	$3, 1^{12}$	14
$8, 5, 1^2$	-40775	$6, 4, 2, 1^3$	-1141	$4^3, 2, 1$	-298	$2^7, 1$	-49
$8, 4, 3$	-38146	$6, 4, 1^5$	-1066	$4^3, 1, 1, 1$	-277	$2^6, 1^3$	-46
$8, 4, 2, 1$	-36715	$6, 3^3$	-1105	$4^2, 3^2, 1$	-226	$2^5, 1^5$	-41
$8, 4, 1^3$	-35602	$6, 3^2, 2, 1$	-1054	$4^2, 3, 2^2$	-210	$2^4, 1^7$	-34
$8, 3^2, 1$	-34961	$6, 3^2, 1^3$	-1015	$4^2, 3, 2, 1^2$	-202	$2^3, 1^9$	-25
$8, 3, 2^2$	-33991	$6, 3, 2^3$	-991	$4^2, 3, 1^4$	-186	$2^2, 1^{11}$	-14
$8, 3, 2, 1^2$	-33373	$6, 3, 2^2, 1^2$	-969	$4^2, 2^3, 1$	-175	$2, 1^{13}$	-1
$8, 3, 1^4$	-32137	$6, 3, 2, 1^4$	-925	$4^2, 2^2, 1^3$	-166	1^{15}	14
$8, 2^3, 1$	-31786	$6, 3, 1^6$	-859	$4^2, 2, 1^5$	-151		
$8, 2^2, 1^3$	-30991	$6, 2^4, 1$	-877	$4^2, 1^7$	-130		
$8, 2, 1^5$	-29666	$6, 2^3, 1^3$	-850	$4, 3^3, 2$	-81		
$8, 1^7$	-27811	$6, 2^2, 1^5$	-805	$4, 3^3, 1^2$	-79		

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